Government of **Western Australia**
School Curriculum and Standards Authority

MATHEMATICS METHODS

Calculator-free

ATAR course examination 2022

Marking key

Marking keys are an explicit statement about what the examining panel expect of candidates when they respond to particular examination items. They help ensure a consistent interpretation of the criteria that guide the awarding of marks.

Section One: Calculator-free **35% (54 Marks)**

Question 1 **(9 marks)**

Consider the derivative function $f'(x) = \dfrac{4x}{x^2 + 3}$.

(a) Determine the rate of change of $f'(x)$ when $x = 1$. (3 marks)

Solution
The second derivative is $$f''(x) = \frac{4(x^2+3) - 4x(2x)}{(x^2+3)^2}$$ Evaluating at $x = 1$ gives $$f''(1) = \frac{(4)(4) - (4)(2)}{(4)^2} = \frac{1}{2}$$
Specific behaviours
✓ correctly differentiates $f'(x)$ ✓ indicates rate of change of $f'(x)$ when $x = 1$ is $f''(1)$ ✓ correctly substitutes into $f''(x)$ and evaluates

(b) Determine $f(x)$ given that $f(1) = \ln(32)$. (4 marks)

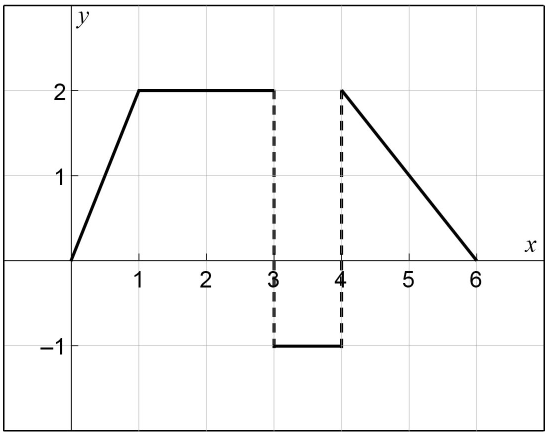
Solution
$$f(x) = \int \frac{4x}{x^2+3}\,dx = 2\int \frac{2x}{x^2+3}\,dx = 2\ln(x^2+3) + c$$ Substituting $f(1) = \ln(32)$ gives $$\ln(32) = 2\ln(1^2+3) + c$$ $$\Rightarrow c = \ln(32) - 2\ln(4) = \ln(32) - \ln(16) = \ln\left(\frac{32}{16}\right) = \ln(2)$$ Hence $$f(x) = 2\ln(x^2+3) + \ln(2)$$
Specific behaviours
✓ integrates correctly ✓ substitutes $f(1) = \ln(32)$ ✓ correctly applies log laws to simplify ✓ evaluates c and states equation for $f(x)$

(c) Determine $\frac{d}{dt}\int_t^3 f(x)\,dx$. (2 marks)

Solution
$\frac{d}{dt}\int_t^3 f(x)\,dx = -\frac{d}{dt}\int_3^t f(x)\,dx$ $= -f(t)$ $= -2\ln(t^2+3) - \ln(2)$
Specific behaviours
✓ uses properties of integrals to reverse integration bounds   ✓ applies the fundamental theorem of calculus to obtain answer

Question 2 **(6 marks)**

Consider the function $f(x)$ shown below.



Evaluate the following integrals.

(a) $\int_0^6 f(x)dx$ (2 marks)

Solution
Summing signed areas $$\int_0^6 f(x)dx = 1 + 4 - 1 + 2 = 6$$
Specific behaviours
✓ expresses integral in terms of signed areas ✓ evaluates integral correctly

(b) $\int_0^4 f(x) - 2\,dx$ (2 marks)

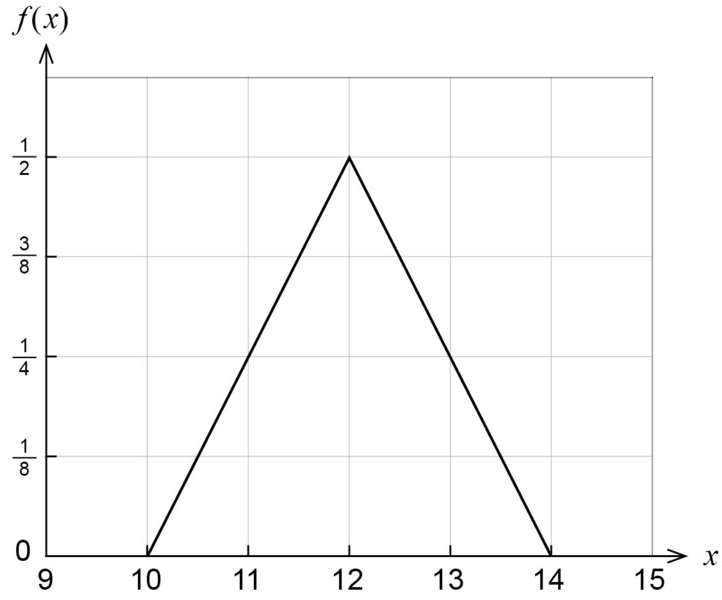
Solution
$$\int_0^4 f(x) - 2\,dx = \int_0^4 f(x)\,dx - \int_0^4 2\,dx$$ $$= 4 - 2 \times 4$$ $$= -4$$
Specific behaviours
✓ expresses integral as the difference between $\int_0^4 f(x)\,dx$ and $\int_0^4 2\,dx$ ✓ evaluates integral correctly

(c) $\int_4^6 f'(x)dx$ (2 marks)

Solution
$$\int_4^6 f'(x)dx = f(6) - f(4) = 0 - 2 = -2$$
Specific behaviours
✓ applies fundamental theorem of calculus ✓ evaluates correctly
Note: Accept other answers that apply the fundamental theorem of calculus.

Question 3 **(11 marks)**

Arnold would like to purchase a toy for his child's birthday. The Isosceles Toy Company claims that the number of weeks until delivery, X, is a random variable whose probability density function is displayed in the graph below.



(a) What is the expected time for the toy to be delivered? (1 mark)

Solution
$E(X) = 12$ weeks
Specific behaviours
✓ states correct expected time, including units

His child's birthday is 13 weeks away.

(b) What is the probability that the Isosceles Toy Company will deliver the toy in time for his child's birthday? (2 marks)

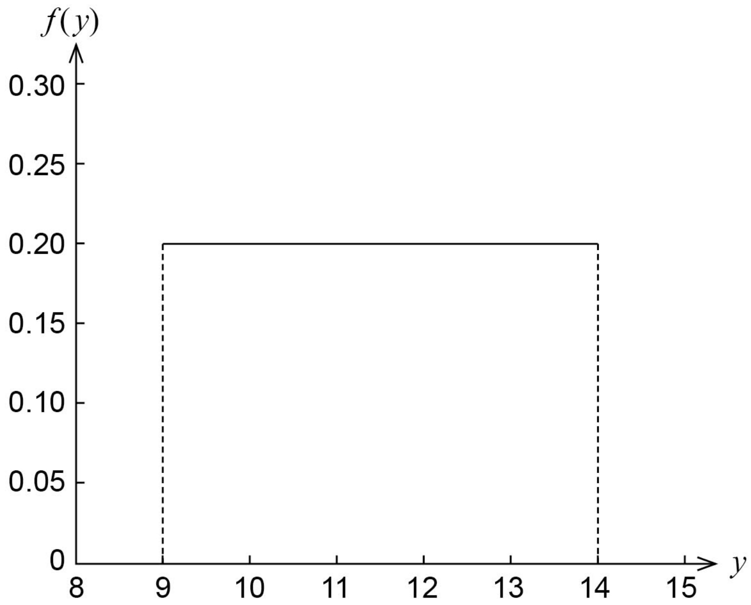
Solution
$P(X < 13) = 1 - \frac{1}{2} \times 1 \times \frac{1}{4}$ $= 1 - \frac{1}{8}$ $= \frac{7}{8}$
Specific behaviours
✓ identifies the probability as the area under the curve between 10 and 13 (or 1 minus the area under the curve between 13 and 14)   ✓ calculates the correct probability

Question 3 (continued)

(c) Given that the toy arrives in time for his child's birthday, what is the probability that it arrives at least one week early? (2 marks)

Solution
$P(X < 12 \mid X < 13) = \dfrac{P(X < 12)}{P(X < 13)}$ $= \dfrac{\frac{1}{2}}{\frac{7}{8}}$ $= \dfrac{4}{7}$
Specific behaviours
✓ determines correct conditional probability statement   ✓ obtains correct probability

Uniform Toys, a rival toy company, claims that the number of weeks until delivery of the same toy, Y, is a random variable whose distribution is displayed in the graph below.



(d) Which toy company should Arnold choose if he would like to maximise the chance that the toy will be delivered in time for his child's birthday? Why? (2 marks)

Solution
$P(Y < 13) = 0.8$ $0.8 < \dfrac{7}{8}$ Arnold should choose the Isosceles Toy Company because the probability of receiving the toy on time from the Isosceles Toy Company is greater than the probability of receiving the toy on time from Uniform Toys.
Specific behaviours
✓ determines the probability of delivering the toy within 13 weeks   ✓ chooses correct toy company and provides justification

Suppose that five people order the toy from Uniform Toys and let Z be a random variable that denotes the number of those people who receive the toy within 13 weeks.

(e) State the distribution for Z. (2 marks)

Solution
$Z \sim \mathrm{Bin}(5, 0.8)$
Specific behaviours
✓ states distribution is binomial ✓ gives correct values for parameters n and p

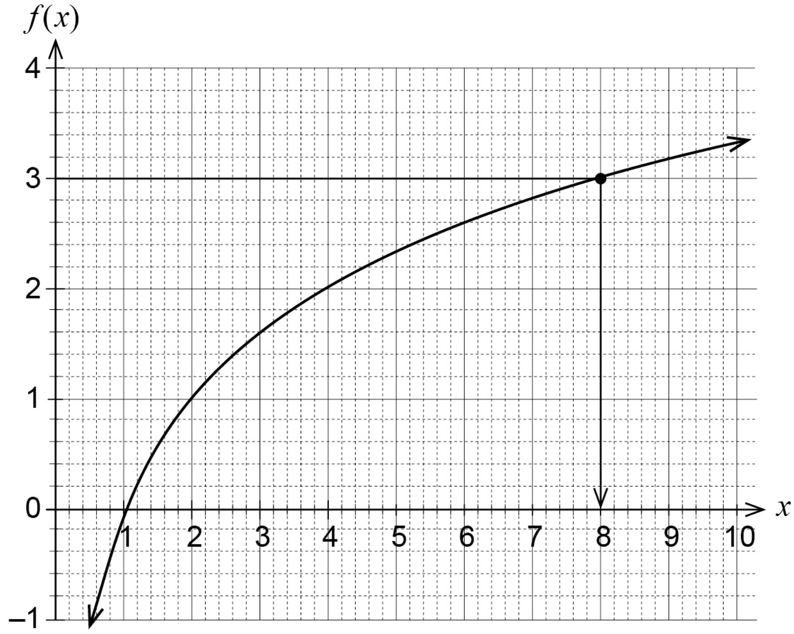
(f) What is the probability that four out of the five people receive the toy within 13 weeks? (2 marks)

Solution
$P(Z = 4) = \binom{5}{4} \times \left(\frac{4}{5}\right)^4 \times \frac{1}{5}$ $= 5 \times \frac{256}{625} \times \frac{1}{5}$ $= \frac{256}{625}$
Specific behaviours
✓ writes correct expression for probability ✓ calculates correct probability

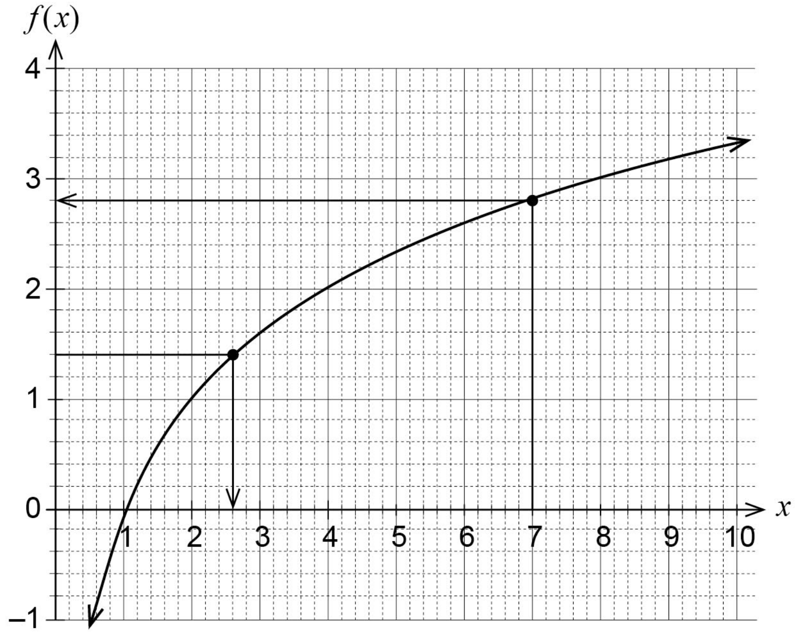
Question 4 **(12 marks)**

(a) Using the graph:

(i) solve $\log_2(x-5)=3$. (2 marks)

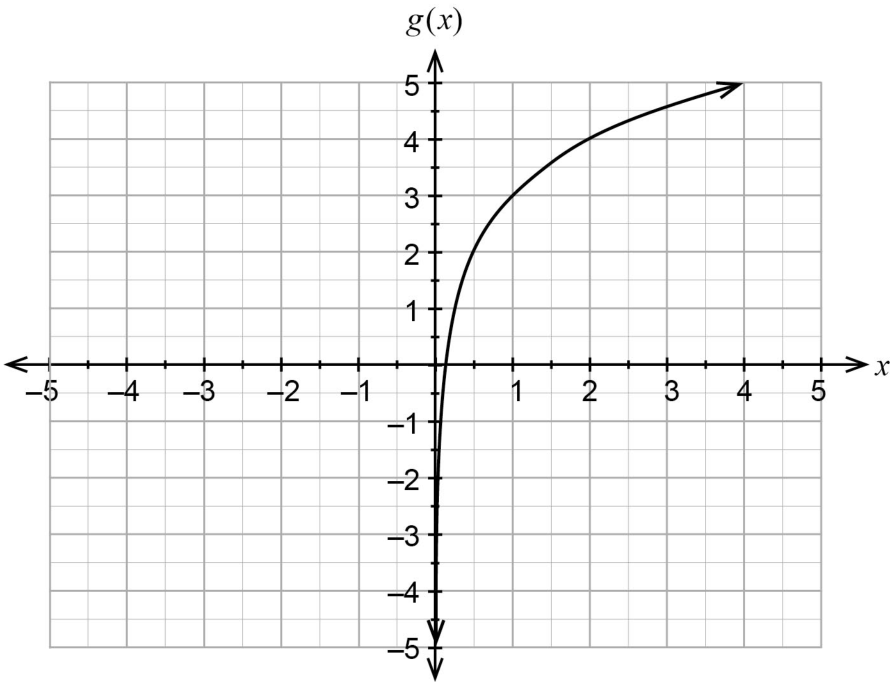
Solution

$\log_2(x-5)=3$ $\log_2 8=3$ (from graph) $x-5=8$ $x=13$
Specific behaviours
✓ identifies $\log_2 8=3$ ✓ solves correctly for x

(ii) determine $\sqrt{7}$, correct to one decimal place. (Hint: let $x = \sqrt{7}$.) (3 marks)

Solution
 $\log_2(x) = \log_2(\sqrt{7})$ $= \log_2\left(7^{\frac{1}{2}}\right)$ $= \frac{1}{2}\log_2(7)$ $\approx \frac{1}{2} \times 2.8$ $= 1.4$ So $\log_2(x) = 1.4$ from graph $x \approx 2.6$ Hence $\sqrt{7} \approx 2.6$
Specific behaviours
✓ applies log laws to express $\log_2(x)$ as $\frac{1}{2}\log_2(7)$ ✓ approximates $\log_2(7)$ as 2.8 ✓ obtains correct approximation to $\sqrt{7}$

Question 4 (continued)

(b) The function $f(x) = \log_2(x)$ is translated to give the new function $g(x)$, which is shown in the graph below.



Determine the equation for $g(x)$. (2 marks)

Solution
$g(x) = f(x) + 3$ $= \log_2(x) + 3$
Specific behaviours
✓ identifies a vertical translation of 3 units up ✓ states correct equation for $g(x)$

(c) (i) Show that $\log_2\left(\frac{1}{x-1}\right) = -\log_2(x-1)$. (2 marks)

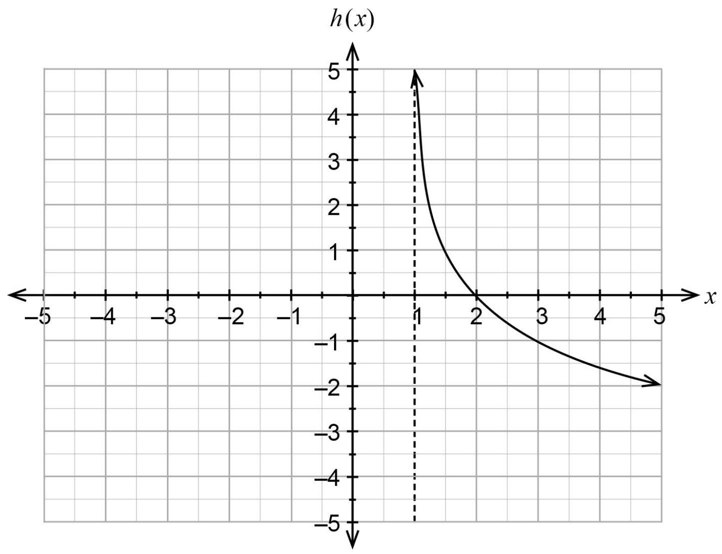
Solution
$\log_2\left(\frac{1}{x-1}\right)$ $= \log_2 1 - \log_2(x-1)$ $= -\log_2(x-1)$
Specific behaviours
✓ applies log laws to obtain $\log_2 1 - \log_2(x-1)$ ✓ recognises $\log_2 1 = 0$ and hence obtains required expression

Alternate solution
$\log_2\left(\frac{1}{x-1}\right)$ $= \log_2(x-1)^{-1}$ $= -\log_2(x-1)$
Specific behaviours
✓ recognises $\frac{1}{x-1}$ is equivalent to $(x-1)^{-1}$ ✓ applies log laws to obtain required expression

(ii) Hence sketch the graph of $h(x) = \log_2\left(\frac{1}{x-1}\right)$ on the axes below. (3 marks)

Solution

$h(x)$ is obtained by reflecting $f(x)$ vertically about the x-axis, and translating 1 unit to the right.



Specific behaviours

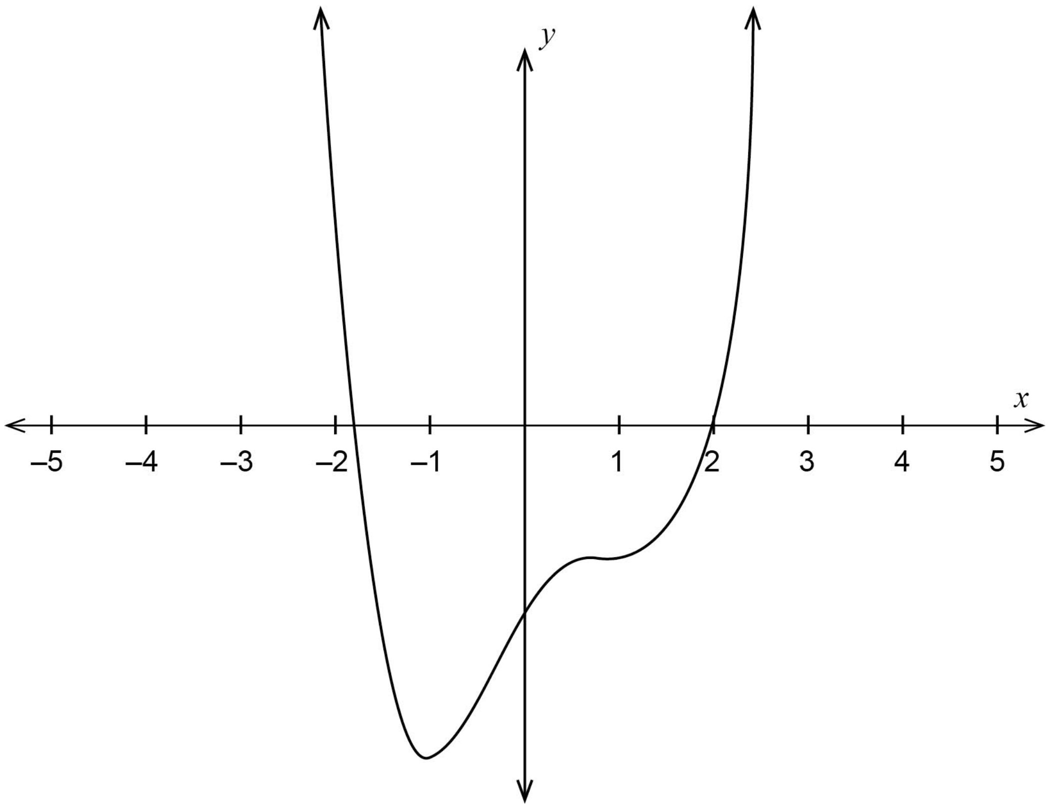
✓ draws asymptote at $x = 1$
✓ graph passes through $(2, 0)$
✓ graph passes through $(3, -1)$ and has correct shape

Question 5 **(5 marks)**

A continuous function, f, satisfies the following conditions:

- $f(2) = 0$
- f has exactly 2 stationary points
- $f'(-1) = 0$ and $f'(1) = 0$
- $f''(-1) = 4$
- $f'(2) > 0$.

Sketch the function on the axes below.

Solution

Specific behaviours
✓ locates an intercept at $x = 2$ ✓ indicates a positive gradient for $x > 1$ ✓ indicates a local minimum at $x = -1$ ✓ locates a horizontal point of inflection at $x = 1$ ✓ sketches a continuous curve with correct shape

Question 6 **(11 marks)**

The table of values below may be used to assist you in answering part (b) of this question.

$\sin(0) = 0$	$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$	$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$	$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$	$\sin\left(\frac{\pi}{2}\right) = 1$
$\cos(0) = 1$	$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$	$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$	$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$	$\cos\left(\frac{\pi}{2}\right) = 0$

(a) (i) Determine $\frac{d}{dx}\left(x\sin\left(\frac{\pi x}{4}\right)\right)$. (2 marks)

Solution
Using the product rule $\frac{d}{dx}x\sin\left(\frac{\pi x}{4}\right) = \sin\left(\frac{\pi x}{4}\right) + \frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right)$
Specific behaviours
✓ applies product rule to evaluate derivative ✓ obtains correct answer

(ii) Hence show that

$$\int \frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right)dx = x\sin\left(\frac{\pi x}{4}\right) + \frac{4}{\pi}\cos\left(\frac{\pi x}{4}\right) + c$$

where c is a constant. (3 marks)

Solution
From part (i) $\frac{d}{dx}x\sin\left(\frac{\pi x}{4}\right) = \sin\left(\frac{\pi x}{4}\right) + \frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right)$ $\Rightarrow \int \frac{d}{dx}x\sin\left(\frac{\pi x}{4}\right)dx = \int \sin\left(\frac{\pi x}{4}\right)dx + \int \frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right)dx$ $\Rightarrow x\sin\left(\frac{\pi x}{4}\right) = -\frac{4}{\pi}\cos\left(\frac{\pi x}{4}\right) + \int \frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right)dx$ $\Rightarrow \int \frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right)dx = x\sin\left(\frac{\pi x}{4}\right) + \frac{4}{\pi}\cos\left(\frac{\pi x}{4}\right) + c$
Specific behaviours
✓ integrates both sides of the result from (i) and correctly evaluates $\int \sin\left(\frac{\pi x}{4}\right)dx$ ✓ applies the fundamental theorem of calculus to evaluate $\int \frac{d}{dx}\left(x\sin\left(\frac{\pi x}{4}\right)\right)dx$ ✓ applies valid mathematical operations to obtain required expression

Question 6 (continued)

(b) The time in minutes, T, between incoming phone calls at a call centre is a random variable with probability density function

$$p(t) = \begin{cases} \dfrac{\pi}{4}\cos\left(\dfrac{\pi t}{4}\right), & 0 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}$$

(i) Determine the probability that the time between two consecutive phone calls is less than 40 seconds. State your answer exactly. (3 marks)

Solution
$P\left(T < \frac{2}{3}\right) = \int_0^{\frac{2}{3}} \frac{\pi}{4}\cos\left(\frac{\pi t}{4}\right) dt$ $= \left[\sin\left(\frac{\pi t}{4}\right)\right]_0^{\frac{2}{3}}$ $= \sin\left(\frac{\pi}{6}\right)$ $= \frac{1}{2}$
Specific behaviours
✓ writes correct integral (recognises that 40 seconds is $t = \frac{2}{3}$ minutes)   ✓ anti-differentiates correctly   ✓ obtains correct answer

(ii) Use the result from part (a)(ii) to determine the expected time between consecutive phone calls. (3 marks)

Solution
The expected value of T is given by $E(T) = \int_0^2 \frac{\pi}{4} t\cos\left(\frac{\pi t}{4}\right) dt$ $= \left[t\sin\left(\frac{\pi t}{4}\right) + \frac{4}{\pi}\cos\left(\frac{\pi t}{4}\right)\right]_0^2$ $= 2\sin\left(\frac{\pi}{2}\right) + \frac{4}{\pi}\cos\left(\frac{\pi}{2}\right) - \frac{4}{\pi}\cos(0)$ $= 2 - \frac{4}{\pi}$
Specific behaviours
✓ writes correct integral expression for the expected value of T (including bounds)   ✓ applies fundamental theorem of calculus to evaluate definite integral   ✓ correctly simplifies answer

An *Acknowledgements variation* document is available on the Authority website.

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